SIMPLE S_r-HOMOTOPY TYPES OF HOM COMPLEXES AND BOX COMPLEXES ASSOCIATED TO r-GRAPHS

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ABSTRACT. For a pair (H_1, H_2) of graphs, Lovász introduced a polytopal complex called the Hom complex $\mathrm{Hom}(H_1, H_2)$, in order to estimate topological lower bounds for chromatic numbers of graphs. The definition is generalized to hypergraphs. Denoted by K_r^r the complete r-graph on r vertices. Given an r-graph H, we compare $\mathrm{Hom}(K_r^r, H)$ with the box complex $\mathsf{B}_{\mathrm{edge}}(H)$, invented by Alon, Frankl and Lovász. We verify that $\mathrm{Hom}(K_r^r, H)$ and $\mathsf{B}_{\mathrm{edge}}(H)$, both are equipped with right actions of the symmetric group on r letters S_r, are of the same simple S_r-homotopy type.

1. INTRODUCTION

In this paper, we consider homotopy types of cell complexes associated to r-graphs which are introduced in order to solve the problem on their chromatic numbers. The idea of assigning a cell complex to graphs was due to Lovász in [Lov78] in his proof of the Kneser's conjecture [Kne56]. To a graph G, Lovász assigned a simplicial complex $\mathsf{N}(G)$, called the *neighborhood complex*. By using its topological property, that is to say, the k-connectivity of $\mathsf{N}(G)$, he succeeded in discovering a new lower bound for the chromatic number of G.

In the case of hypergraphs, the first topological lower bound for the chromatic number of an r-graph was derived by a simplicial complex $\mathsf{B}_{\mathrm{edge}}(G)$ called the box complex, which was invented by Alon, Frankl and Lovász [AFL86]. It also played an important role in a proof of the Erdős' conjecture [Erd76], which is a generalization of Kneser's conjecture.

Lovász also introduced a polytopal complex associated to a pair (G, H) of graphs, called the Hom complex $\mathrm{Hom}(G, H)$. It is a generalization of $\mathsf{N}(H)$ in view of $\mathrm{Hom}(K_2, H)$ and $\mathsf{N}(H)$ having the same homotopy type [Koz06]. Here K_2 denotes the complete graph on 2 vertices. There are also many researches on the homotopy type of $\mathrm{Hom}(K_2, H)$, comparing with other simplicial complexes constructed for (hyper)graph coloring problems such as $\mathsf{B}_{\mathrm{chain}}(G)$ by Kříž [Kří92] or $\mathsf{B}(G), \mathsf{B}_0(G)$ by Matoušek and Ziegler [MZ04]. However, there are still no results in the case of r-graphs. The motivation of this research is to find an r-graph which generalizes the results to the case of r-graphs.

The construction of the Hom complex is also extended to hypergraphs by Kozlov in [Koz07]. We notice here that the complete r-graph on r vertices K_r^r is the only r-graph having one edge as K_2, and that both $\mathrm{Hom}(K_r^r, H)$ and $\mathsf{B}_{\mathrm{edge}}(H)$ are equipped with right actions of the symmetric group on r letters S_r. We obtain the following result on equivariant simple homotopy types by making use of equivariant acyclic partial matchings:

Theorem (Theorem 4.11)**.** For any r-graph H, the Hom complex $\mathrm{Hom}(K_r^r, H)$ and the box complex $\mathsf{B}_{\mathrm{edge}}(H)$ have the same simple S_r-homotopy type.

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2. Preliminaries

In this section, we collect some definitions which are needed in our arguments. First, we write $[k]$ as the set $\{0, 1, \ldots, k\}$.

***r*-graphs.** A *hypergraph* is a triple $H = (V(H), E(H), \varepsilon_H)$ of sets $V(H), E(H)$ and a map $\varepsilon_H : E(H) \to \coprod_{r\geq 1}(V(H)^r/S_r)$. Here S_r is the symmetric group on r-letters acting on $V(H)^r$ by permutation. Given two hypergraphs H_1 and H_2, a *hypergraph homomorphism* is a pair (f_V, F_E) of $f_V : V(H_1) \to V(H_2)$ and $f_E : E(H_1) \to E(H_2)$ satisfying the following commutative diagram:

$$\begin{tikzcd} E(H_1) \arrow[r, "\varepsilon_{H_1}"] \arrow[d, "f_E"'] & \coprod_{r\geq 1}(V(H_1)^r/S_r) \arrow[d, "\tilde{f}_V"] \\ E(H_2) \arrow[r, "\varepsilon_{H_2}"'] & \coprod_{r\geq 1}(V(H_2)^r/S_r), \end{tikzcd}$$

where $\tilde{f}_V$ is the map induced by f_V. Then, we obtain the category $\mathbf{H}$ of hypergraphs and hypergraph homomorphisms.

We denote here an equivalence class $[v_0, v_1, \ldots, v_{r-1}] \in V(H)^r/S_r$ simply by $v_0v_1\ldots v_{r-1}$. A hypergraph H is *r-uniform* if $\operatorname{Im}\varepsilon_H \subset V(H)^r/S_r$. H is *simple* if ε_H is injective. Moreover, H is *nondegenerate* if

$$\operatorname{Im}\varepsilon_H \subset \coprod_{r\geq 1}\left\{v_0\cdots v_{r-1} \in V(H)^r/S_r \,\middle|\, v_i \neq v_j \text{ whenever } i \neq j\right\}.$$

For simplicity, simple nondegenerate r-uniform hypergraphs are called *r-graphs*. Denoted by $\mathbf{H}_r$ the full subcategory of $\mathbf{H}$ consisting of r-graphs. For example, the complete r-graph on m vertices, denoted by K^r_m, is an r-graph with $|V(K^r_m)| = m$ and $\varepsilon_{K^r_m}$ being bijective. Since the map ε_H of an r-graph H is a bijection $E(H) \to \operatorname{Im}\varepsilon_H$, for simply, we identify $E(H)$ with $\operatorname{Im}\varepsilon_H$, and write, for example, $v_0\ldots v_{r-1} \in E(H)$.

The category $\mathbf{C}$-G. Let G be a group. Denoted by G^{op} the group whose elements are elements of G and multiplication defined by gh (in G^{op}) $= hg$ (in G). For an object X of a category $\mathbf{C}$, a *right action* of G on X is a homomorphism $\rho : G^{op} \to \operatorname{Hom}_{\mathbf{C}}(X, X)$. We denote by $\mathbf{C}$-G the category whose objects are all pairs (X, ρ) of object X of $\mathbf{C}$ and a right action ρ. A morphism from (X_1, ρ_1) to (X_2, ρ_2) is a morphism $f \in \operatorname{Hom}_{\mathbf{C}}(X_1, X_2)$ such that $f \circ \rho_1(g) = \rho_2(g) \circ f$ for any $g \in G^{op}$. We note here that, for two categories $\mathbf{C}$ and $\mathbf{D}$, a functor $F : \mathbf{C} \to \mathbf{D}$ induces a functor F-$G : \mathbf{C}$-$G \to \mathbf{D}$-G.

Simplicial complexes and polytopal complexes. An *(abstract) simplicial complex* is a pair (V, K) of a set V and a collection K of subsets of V closed under taking subsets. We denote a simplicial complex (V, K) briefly by K and write V as $V(\mathsf{K})$. Each elements in K is called a *simplex* or a *cell* of K. If $F \in \mathsf{K}$ and $F' \subset F$, we say that F' is a *face* of F, and, at the same time, F is a *coface* of F'. A *subcomplex* of K is a simplicial complex K' such that $F \in \mathsf{K}'$ implies that $F \in \mathsf{K}$.

For two simplicial complex K and K', a *simplicial map* $f : \mathsf{K} \to \mathsf{K}'$ is a map $f : V(\mathsf{K}) \to V(\mathsf{K}')$ satisfying that $f(F) \in \mathsf{K}'$ if $F \in \mathsf{K}$. Let **ASC** denote the category of simplicial complexes and simplicial maps. In particular, an object in the category **ASC**-G is called a *simplicial G-complex*.

Let P be a convex polytope. A *proper face* of P is of the form $\operatorname{conv}(V(P) \cap h)$, where h is a hyperplane satisfying $(\operatorname{Int} P) \cap h = \varnothing$ and $V(P)$ denotes the vertex set of P. The term "coface" for convex polytopes is also defined analogously. Note here that the empty set is also a proper face of any polytopes.

A *polytopal complex* is a collection K of convex polytopes in some $\mathbb{R}^N$ satisfying that (1) every face of $P \in \mathsf{K}$ is also in K, and (2) the intersection of $P_1, P_2 \in \mathsf{K}$ is a face of both. Elements in K are

called *cells* of K. The *underlying space* of a polytopal complex K is the subspace of $\mathbb{R}^N$ defined by $|\mathsf{K}| = \bigcup_{P\in\mathsf{K}} P$. A *subcomplex* of K is a subcollection K' of K which is itself a polytopal complex.

For two polytopal complexes K_1 and K_2, a *polytopal map* $f : \mathsf{K}_1 \to \mathsf{K}_2$ is a map $f : |\mathsf{K}_1| \to |\mathsf{K}_2|$ satisfying that the restrictions $f|_F$ to each $F \in \mathsf{K}_1$ is affine. Moreover, a polytopal map $f : \mathsf{K}_1 \to \mathsf{K}_2$ is said to be *regular* if $F \in \mathsf{K}_1$ implies that $f(F) \in \mathsf{K}_2$. In this paper, we will make use of the category **PTC** consisting of polytopal complexes and polytopal maps and its subcategory $\mathbf{PTC}_{\mathrm{reg}}$ consisting of polytopal complexes and regular polytopal maps. An object of the category **PTC**-G (or $\mathbf{PTC}_{\mathrm{reg}}$-$G$) is called a polytopal G-complex.

Posets. Let **Poset** denote the category of posets and poset maps (i.e. a map $f : P \to Q$ satisfying $f(x) \le_Q f(y)$ whenever $x \le_P y$). An object in the category **Poset**-G is called a G-poset.

Given a poset P, we call a totally ordered subset $A = \{A_0, A_1, \dots, A_k\}$, where each $A_i \in P$ and $A_0 <_P A_1 <_P \cdots <_P A_{k-1}$ a *k-chain* in P. The number k is called the *length* of A, denoted by $\#A$. In this paper, elements in a chain A in P are written by A_i ($i \in [\#A]$). The *order complex* of P, denoted by $\Delta(P)$, is the simplicial complex on P whose k-simplices are the k-chains in P. A poset map $f : P \to Q$ induces a simplicial map $\Delta(f) : \Delta(P) \to \Delta(Q)$, and so $\Delta(\cdot)$ is a covariant functor **Poset** $\to$ **ASC**.

The *face poset* of a simplicial (polytopal) complex K, denoted by $\mathcal{F}(\mathsf{K})$, is a poset of all nonempty cells of K ordered by inclusion. Each simplicial (polytopal) map $f : \mathsf{K} \to \mathsf{K}'$ induces a poset map $\mathcal{F}(f) : \mathcal{F}(\mathsf{K}) \to \mathcal{F}(\mathsf{K}')$. So we obtain covariant functors $\mathcal{F}(\cdot)$: **ASC** $\to$ **Poset** and $\mathcal{F}(\cdot)$: **PTC** $\to$ **Poset**

For $x, y \in P$, we call *x covers y*, and write $x \succ y$, if $y <_P x$ and there is no $z \in P$ such that $y <_P z <_P x$.

3. Equivariant simple homotopy types

Now let K be a simplicial or a polytopal complex. Maximal cells of K are called *facets*. A cell $\sigma \in \mathsf{K}$ is *free* if σ is a proper face of only one facet $\varphi_\sigma \in \mathsf{K}$. A collection $\mathcal{F}$ of free cells of K is said to be *independently free* if, for any σ, $\sigma' \in \mathcal{F}$, $\sigma \neq \sigma'$ implies that there is no cell in K which is a coface of both σ and σ'.

The *deletion* of a cell $F \in \mathsf{K}$, denoted by $\mathrm{dl}_F(\mathsf{K})$, is the subcomplex of K consisting of all $F' \in \mathsf{K}$ such that F is not a face of F'. We also define the deletion $\mathrm{dl}_S(\mathsf{K})$ of a set S of cells of K from K as the intersection of $\mathrm{dl}_F(\mathsf{K})$ over all $F \in S$.

Now we define the notion of G-collapsings, following Larrión et. al. in [LPVF08]. Note here that, for a simplicial (polytopal) G-complex K, the orbit σG of a free cell $\sigma \in \mathsf{K}$ is a collection of free cells in K. Let σ be a free cell of K with $\dim \varphi_\sigma = \dim \sigma + 1$. Suppose σG being independently free. An *elementary G-collapsing* of K with respect to σ is defined as the process to obtain $\mathrm{dl}_{\sigma G}(\mathsf{K})$ from K. Conversely, an *elementary G-expanding* of K with respect to σ is defined to be the process to obtain K from $\mathrm{dl}_{\sigma G}(\mathsf{K})$.

We denote by $\mathsf{K} \searrow_G \mathsf{K}'$ if there exists an elementary G-collapsing of K onto its G-subcomplex K'. Moreover, we say that K *G-collapses* onto a G-subcomplex K' if there is a sequence of elementary G-collapsings leading from K to K'. Two simplicial (polytopal) G-complex K and L are said to have the same *simple G-homotopy type* if there is a sequence of elementary G-collapsings and elementary G-expandings leading from K to L. Such a sequence is called a *formal G-deformation*.

3.1. Simple G-homotopy types of subdivisions.

It is well-known on a relationship between a simplicial (polytopal) complex K and its barycentric subdivision $\mathrm{sd}\,\mathsf{K}$ that they are of the same simple homotopy type. However, we need an equivariant version of this result in our argument.

Following the construction of a formal deformation by Kozlov in [Koz06], it is useful to define an equivariant stellar subdivision of K.

Definition 3.1. Let K be a simplicial G-complex and σ be a simplex of K such that, in σG, $g \neq g'$ implies that no simplex in K being a coface of both σg and $\sigma g'$. The *stellar G-subdivision* of K at the orbit σG, denoted by $\mathrm{sd}(\mathsf{K}, \sigma G)$, is the simplicial G-complex on $V(\mathsf{K}) \coprod \sigma G$ with the following set of simplices:

$$\mathrm{sd}(\mathsf{K}, \sigma G) = \bigcap_{g \in G} \{F \in \mathsf{K} \mid \sigma g \text{ is not a face of } F\}$$
$$\cup \bigcup_{g \in G} \{F \coprod \{\sigma g\} \mid F \in \mathsf{K},\ \sigma g \text{ is not a face of } F, \text{ and } \sigma g \cup F \in \mathsf{K}\}.$$

We can define the stellar subdivision for a polytopal G-complex K analogously by replacing elements in σG with their barycenters.

Making use of stellar G-subdivisions, we obtain our desired result:

Proposition 3.2. Let K be a simplicial or polytopal G-complex. Then K and its barycentric subdivision $\mathrm{sd}\,\mathsf{K}$ have the same simple G-homotopy type.

Proof. Choose a cell σ from each orbit such that they preserves inclusion order in $\mathcal{F}(\mathsf{K})$ and construct a totally ordered set L of these σ's, such that $\bigcup_{\sigma \in L} \sigma G = \mathcal{F}(\mathsf{K})$ as sets. Then a simplicial G-complex obtained by a sequence of stellar G-subdivisions of K at the orbits of simplices in decreasing order with respect to L is isomorphic to $\mathrm{sd}\,\mathsf{K}$. Hence, it suffices to consider a formal deformation leading from K to the stellar G-subdivision $\mathrm{sd}(\mathsf{K}, \sigma G)$ at the orbit of the maximum cell $\sigma \in L$.

First, add cones over each $\mathrm{st}_{\mathsf{K}}(\sigma g)$, $g \in G$. This construction implies that, for each face σ' of σ, $\sigma' G$ is a collection of free cells which is independently free. Hence, we obtain a sequence of elementary G-expandings leading to cones. Here we obtain the unique facet containing $\sigma g \in \sigma G$ in each added cone. Then we obtain our desired result by taking an elementary G-collapsing with respect to σG. □

4. Hom complexes

The construction the Hom complexes was extended to hypergraphs by Kozlov [Koz07]. In this paper, however, we will consider only the one associated to a pair of r-graphs.

Definition 4.1. Let H_1, H_2 be r-graphs. A map $f : V(H_1) \to 2^{V(H_2)} \setminus \{\emptyset\}$ is called a *hypergraph multihomomorphism* if every map $f_0 : V(H_1) \to V(H_2)$ satisfying $f_0(v) \in f(v)$ for any $v \in V(H_1)$ induces a hypergraph homomorphism.

For hypergraphs H_1, H_2, we write P_{H_1,H_2} as the poset of all hypergraph multihomomorphisms ordered by $f \leq g$ if and only if $f(v) \subset g(v)$ for any $v \in V(H_1)$. The Hom complex $\mathrm{Hom}(H_1, H_2)$ is construced from this poset as follows:

Definition 4.2. Let H_1, H_2 be r-graphs. The *Hom complex* is the polytopal complex

$$\mathrm{Hom}(H_1, H_2) = \left\{ \prod_{v \in V(H_1)} \Delta^{f(v)} \right\}_{f \in P_{H_1,H_2}}.$$

Here Δ^S denotes a simplex with the vertex set S.

Denoted by $\mathbf{H}_r^i$ a subcategory of $\mathbf{H}_r$ consisting of r-graphs and injective hypergraph homomorphisms. By definition, we obtain the following commutative diagrams concerning functorial properties:

$$\begin{tikzcd}
\mathbf{H}_r \arrow[r, "P_{H,-}"] & \mathbf{Poset} \\
 & \mathbf{PTC} \arrow[u, "\mathcal{F}"'] \\
\mathbf{H}_r^i \arrow[uu, hook] \arrow[r, "\mathrm{Hom}(H,-)"'] & \mathbf{PTC}_{\mathrm{reg}} \arrow[u, hook]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{H}_r^{op} \arrow[r, "P_{-,H}"] \arrow[dr, "\mathrm{Hom}(-{,}H)"'] & \mathbf{Poset} \\
 & \mathbf{PTC} \arrow[u, "\mathcal{F}"'] \\
(\mathbf{H}^i)_r^{op} \arrow[uu, hook] \arrow[r, "\mathrm{Hom}(-{,}H)"'] & \mathbf{PTC}_{\mathrm{reg}} \arrow[u, hook]
\end{tikzcd}$$

In particular, we obtain right $\mathrm{Aut}(H_1)$-actions on both the poset P_{H_1,H_2} and the polytopal complex $\mathrm{Hom}(H_1, H_2)$. Furthermore, we can see that $f < g$ in P_{H_1,H_2} if and only if $\prod_{v\in V(H_1)} \Delta^{f(v)}$ is a proper face of $\prod_{v\in V(H_1)} \Delta^{g(v)}$. Therefore, $\mathcal{F}(\mathrm{Hom}(H_1, H_2))$ and P_{H_1,H_2} are $\mathrm{Aut}(H_1)$-isomorphic as posets, and $\mathrm{sd}\,\mathrm{Hom}(H_1, H_2)$ and $\Delta(P_{H_1,H_2})$ are $\mathrm{Aut}(H_1)$-isomorphic as simplicial complexes.

4.1. **Comparison between Hom complexes and box complexes.** Let (G, H) be a pair of r-graphs. As stated before, we are interested in homotopy type of the Hom complex $\mathrm{Hom}(G.H)$, comparing with simplicial complexes associated to an r-graph H. We now give the definition of the box complex $\mathsf{B}_{\mathrm{edge}}(H)$ invented by Alon, Frankl and Lovász in [AFL86]:

Recall that the collection $\{A_j\}_{j=0}^{r-1}$ of subsets of $V(H)$ *generates the complete r-partite sub-r-graph in H* if, for any $x_j \in A_j,\ j \in [r-1]$, $x_0x_1\cdots x_{r-1}$ is an edge of H. In particular, if $V(H) = \bigcup_{j=1}^r A_j$, H itself is said to be the *complete r-partite r-graph*, denoted by $K^r_{m_0,\dots,m_{r-1}}$ if $|A_j| = m_j,\ j \in [r-1]$.

Definition 4.3 (See [AFL86])**.** Let H be an r-graph. A simplicial complex $\mathsf{B}_{\mathrm{edge}}(H)$ is defined to be a pair $(V, \mathsf{B}_{\mathrm{edge}}(H))$ of the vertex set V consisting of all $(v_1, \dots, v_r) \in V(H)^r$ such that $v_1\cdots v_r \in E(H)$ and the set of simplices $B_{\mathrm{edge}}(H)$ consisting of all subsets $F \subset V$ such that $\{\mathrm{pr}_j(F)\}_{j=1}^r$ is the collection of pairwise disjoint sets generating the complete r-partite sub-r-graph in H. Here $\mathrm{pr}_j(F)$ denotes the projection of F onto its j-th factor.

Now we consider relationships between the Hom complexes and the box complexes. As stated before, $\mathrm{Hom}(K_2, H)$ has the same (simple) homotopy type as the neighborhood complex $\mathsf{N}(H)$ and other box complexes. In the case of r-graph, since K_2 has only one edge, we thought that the complete r-graph K_r^r, which also has only one edge, may play an important role in determining homotopy types of the Hom complexes. Thus, we now compare homotopy types between $\mathrm{Hom}(K_r^r, H)$ and $\mathsf{B}_{\mathrm{edge}}(H)$. However, we cannot do it directly because $\mathrm{Hom}(K_r^r, H)$ is a polytopal while $\mathsf{B}_{\mathrm{edge}}(H)$ is a simplicial complex. We consider their face posets and construct two maps between them as follows:

$$p : \mathcal{F}(\mathsf{B}_{\mathrm{edge}}(H)) \to P_{K_r^r,H};\ p(F)(j) = \mathrm{pr}_j(F);$$

$$i : P_{K_r^r,H} \to \mathcal{F}(\mathsf{B}_{\mathrm{edge}}(H));\ i(\varphi) = \prod_{j=1}^r \varphi(j).$$

Notice here that both $\mathrm{Hom}(K_r^r, H)$ and $\mathsf{B}_{\mathrm{edge}}(H)$ are equipped with right S_r-actions. We claim that both p and i are S_r-equivariant poset maps whose composition $p \circ i$ is the identity on $P_{K_r^r,H}$.

Indeed, for the S_r-equivariance of p, given a simplex $S = \{(v_1^j, \dots, v_r^j)\}_{j\in J} \in \mathcal{F}(\mathsf{B}_{\mathrm{edge}}(H))$ and $\sigma \in S_r$, we have $S\sigma = \{(v_{\sigma(1)}^j, \dots, v_{\sigma(r)}^j)\}_{j\in J}$. Recall that the right S_r-action on $P_{K_r^r,H}$ is given as, for $\sigma \in S_r$, $\sigma : P_{K_r^r,H} \to P_{K_r^r,H}$; $\varphi \mapsto \varphi \circ \sigma$. Hence, for all $l \in [r]$,

$$p(S\sigma)(l) = \{v_{\sigma(l)}^j\}_{j\in J} = p(S)(\sigma(l)) = (p(S)\sigma)(l).$$

For the S_r-equivariant of i, given $f \in P_{K_r^r,H}$ and $\sigma \in S_r$, we have

$$\begin{aligned} i(f)\sigma &= \left(\prod_{j=1}^{r} f(j)\right)\sigma \\ &= \{(v_1, \dots, v_r) \mid v_j \in f(j),\ j \in [r]\}\sigma \\ &= \{(v_{\sigma(1)}, \dots, v_{\sigma(r)}) \mid v_{\sigma(j)} \in f(\sigma(j)),\ j \in [r]\} \\ &= \prod_{j=1}^{r} f \circ \sigma(j) = i(f\sigma). \end{aligned}$$

The injectivity of i implies that the order complex $\Delta(i(P_{K_r^r,H}))$, which can be identified with the barycentric subdivision $\mathrm{sd}\,\mathrm{Hom}(K_r^r, H)$, is an S_r-subcomplex of $\mathrm{sd}\,\mathsf{B}_{\mathrm{edge}}(H)$.

Here we remark that, in general, the composition $i \circ p$ may not be the identity, as shown in the following example.

Example 4.4. Consider the complete r-partite r-graph $K_{1,\dots,1,2,2}^r$ generated by the collection

$$\{\{a_0\}, \dots, \{a_{r-3}\}, \{b_1, b_2\}, \{c_1, c_2\}\}.$$

For instance, taking a simplex

$$F = \{(a_0, \dots, a_{r-3}, b_1, c_1), (a_0, \dots, a_{r-3}, b_2, c_2)\} \in \mathcal{F}(\mathsf{B}_{\mathrm{edge}}(K_{1,\dots,1,2,2}^r),$$

we find that

$$\mathrm{pr}_j(F) = \begin{cases} \{a_j\} & \text{if } j \in [r-3] \\ \{b_1, b_2\} & \text{if } j = r-2 \\ \{c_1, c_2\} & \text{if } j = r-1. \end{cases}$$

Hence, $i \circ p(F) \neq F$. With this example, we can conclude that there is an example of r-graph H whose poset $i(P_{K_r^r,H})$ is a proper S_r-subposet of $\mathcal{F}(\mathsf{B}_{\mathrm{edge}}(H))$.

Moreover, we can conclude that $\Delta(i(P_{K_r^r,K_{1,\dots,1,2,2}^r}))$ and $\mathrm{sd}\,B_{\mathrm{edge}}(K_{1,\dots,1,2,2}^r)$ are not isomorphic.

We also introduce an example of r-graph implying that $i \circ p$ being the identity, and hence, two cell complexes are S_r-isomorphic:

Example 4.5. Considering the complete r-partite r-graph $K_{1,\dots,1,n}^r$ ($n \in \mathbb{N}$), we find that each simplex F of $\mathsf{B}_{\mathrm{edge}}(K_{1,\dots,1,n}^r)$ can be written as the product of sets, $r-1$ sets of them having cardinality 1. Therefore, $i \circ p = 1$.

Remark here that the structures of both $\mathrm{Hom}(K_r^r, H)$ and $\mathsf{B}_{\mathrm{edge}}(H)$, associated to an r-graph H, depend on the containment of complete r-partite r-subgraphs in H. If an r-graph H containing $K_{m_1,\dots,m_r}^r$ where $|\{i \mid m_i \geq 2\}| \geq 2$, then it also contains the complete r-partite r-graph $K_{1,\dots,1,2,2}^r$. Together with the above examples, we obtain the following criterion of determining whether the Hom complexes and the box complexes are isomorphic:

Proposition 4.6. Let H be an r-graph. Then $\Delta(i(P_{K_r^r,H})) \cong \mathrm{sd}\,\mathsf{B}_{\mathrm{edge}}(H)$ if and only if H does not contain the complete r-partite sub-r-graph $K_{1,\dots,1,2,2}^r$.

Example 4.7. Note that the complete r-partite r-graph $K^r_{1,\ldots,1,2,2}$ has $r+2$ vertices. Then, for the complete r-graph K^r_n, two simplicial complexes $\Delta(i(P_{K^r_r,K^r_n}))$ and $\operatorname{sd}\mathsf{B}_{\mathrm{edge}}(K^r_n)$ are isomorphic if and only if $n \le r+1$.

4.2. **Simple S_r-homotopy type of $\mathrm{Hom}(K^r_r, H)$ and $\mathsf{B}_{\mathbf{edge}}(H)$.** Now we return to the argument of verifying that $\mathrm{Hom}(K^r_r, H)$ and $\mathsf{B}_{\mathrm{edge}}(H)$ have the same simple homotopy type. Our strategy is to show that

(1) both $\mathrm{Hom}(K^r_r, H)$ and $\mathsf{B}_{\mathrm{edge}}(H)$ have the same simple homotopy type with their barycentric subdivisions, and
(2) $\operatorname{sd}\mathsf{B}_{\mathrm{edge}}(H)$ S_r-collapses onto $\operatorname{sd}\mathrm{Hom}(K^r_r, H)$.

The statements in the first step are proved by Proposition 3.2. To prove the second one, we will verify the existence of S_r-collapsing of $\operatorname{sd}\mathsf{B}_{\mathrm{edge}}(H)$ onto $\Delta(i(P_{K^r_r,H}))$ by making use of an equivariant acyclic partial matching. We give here its definition and its relationships between an equivariant collapsing:

Definition 4.8. Let G be a finite group and K be a simplicial G-complex. A *partial G-matching* on $\mathcal{F}(\mathsf{K})$ is a pair (Σ,μ) of a G-subset Σ of $\mathcal{F}(\mathsf{K})$ and a G-equivariant injection $\mu : \Sigma \to \mathcal{F}(\mathsf{K})\setminus\Sigma$ such that $\mu(x) \succ x$ for any $x \in \Sigma$. Elements in $\mathcal{F}(\mathsf{K}) \setminus (\Sigma \cup \mu(\Sigma))$ are called *critical*. Such a partial G-matching is *acyclic* if there is no sequence of distinct elements $x_0, x_1, \ldots, x_t \in \Sigma$ $(t \ge 1)$ such that

$$\mu(x_0) \succ x_1,\ \mu(x_1) \succ x_2, \ldots,\ \mu(x_{t-1}) \succ x_t \text{ and } \mu(x_t) \succ x_0.$$

Proposition 4.9. Let G be a finite group, K a simplicial G-complex and K' a G-subcomplex of K. Then K G-collapses onto K' if and only if there is an acyclic partial G-matching on $\mathcal{F}(\mathsf{K})$ whose set of critical elements is just $\mathcal{F}(\mathsf{K}')$.

Proof. First, we assume that K G-collapses onto K'. Then we have a sequence of elementary G-collapsings

$$\mathsf{K} = \mathsf{K}_0 \searrow_G \mathsf{K}_1 \searrow_G \mathsf{K}_2 \searrow_G \cdots \searrow_G \mathsf{K}_k = \mathsf{K}';$$

and we can find simplices $\sigma_0, \sigma_1, \ldots, \sigma_k$ in K such that, for each $i \in [k]$, σ_i is free in K_i; $\dim\varphi_{\sigma_i} = \dim\sigma_i + 1$; $\sigma_i G$ is independently free; and $\mathsf{K}_{i+1} = \mathsf{K}_i \setminus (\sigma_i G \cup \varphi_{\sigma_i} G)$. Let $\Sigma = \coprod_{i=0}^k \sigma_i G$; and $\mu : \Sigma \to \mathcal{F}(\mathsf{K}) \setminus \Sigma$ be defined by $\mu(\sigma_i g) = \varphi_{\sigma_i} g$. Then the pair (Σ,μ) is an acyclic partial G-matching of $\mathcal{F}(\mathsf{K})$ whose set of critical elements is $\mathcal{F}(\mathsf{K}')$.

We state here only a proof of μ being injective: note first that, if we let $i < j$, we find that, for any $g, g' \in G$, $\varphi_{\sigma_i} g \notin K_j$ while $\varphi_{\sigma_j} g' \in K_j$, so $\varphi_{\sigma_i} g \ne \varphi_{\sigma_j} g'$. Hence, $\mu(G\sigma_i) \cap \mu(G\sigma_j) = \varnothing$. Then, it suffices to verify the injectivity of each restriction $\mu|_{\sigma_i G}$.

Suppose that there exist $g, g' \in G$ such that $\mu(\sigma_i g) = \mu(\sigma_i g')$, that is, $\varphi_{\sigma_i} g = \varphi_{\sigma_i} g'$. Then, $\varphi_{\sigma_i} g$ is a simplex in K_i containing both $\sigma_i g$ and $\sigma_i g'$. Since $\sigma_i G$ is independently free, we must have $\sigma_i g = \sigma_i g'$.

Let us prove the converse. Let (Σ,μ) be an acyclic G-matching on $\mathcal{F}(\mathsf{K})$ whose set of critical elements is $\mathcal{F}(\mathsf{K}')$. We give here an algorithm to construct K from its subcomplex K'.

Let Q be the set of elements of Σ already added to K' and W the set of minimal elements in $\mathcal{F}(\mathsf{K}) \setminus \mathcal{F}(\mathsf{K}')$. Suppose first $Q = \varnothing$. We can find $\tau \in W$ such that, for any $g \in G$, $\mu(\tau g) = \mu(\tau) g$ is the only simplex covering τg; if not, we can choose elements of W contradicting the assumption that (Σ,μ) is acyclic.

Set $\overline{\mathsf{K}} = \mathsf{K}' \cup \tau G \cup \mu(\tau) G$. This $\overline{\mathsf{K}}$ is a simplicial G-complex: if there were a proper face of τg in $\mathcal{F}(\mathsf{K}) \setminus \mathcal{F}(\mathsf{K}')$, then τg cannot be minimal in $\mathcal{F}(\mathsf{K}) \setminus \mathcal{F}(\mathsf{K}')$, contradicting $\tau g \in W$. Moreover, the orbit τG is a collection of free faces which is independently free: since μ is injective and G-equivariant,

$\tau g \neq \tau g'$ implies that $\mu(\tau)g \neq \mu(\tau)g'$, that is, no facets in $\mathcal{F}(\overline{\mathsf{K}})$ cover both τg and $\tau g'$ if $g \neq g'$. So we can conclude that $\overline{\mathsf{K}}$ elementary G-collapses onto K'.

Delete all elements in τG from W, set $Q := Q \cup \tau G \cup \mu(\tau)G$, $\mathsf{K}' = \overline{\mathsf{K}}$, and repeat our argument until $W = \emptyset$. If $W = \emptyset$, take a new W of minimal elements in $\mathcal{F}(\mathsf{K}) \setminus (\mathcal{F}(\mathsf{K}') \cup Q)$ and continue our argument until $Q = \mathcal{F}(\mathsf{K}) \setminus \mathcal{F}(\mathsf{K}') = \Sigma \cup \mu(\Sigma)$; and we obtain a sequence of elementary G-collapsings leading from K to K'. □

By this proposition, if one wants to verify that two simplicial G-complexes have the same simple homotopy types, it suffices to construct an acyclic partial G-matching on their face posets. Now we give a construction for our main result:

Lemma 4.10. For an r-graph H, $\mathrm{sd}\,\mathsf{B}_{\mathrm{edge}}(H)$ S_r-collapses onto $\Delta(i(P_{K_r^r,H}))$.

Proof. Since $\Delta(i(P_{K_r^r,H}))$ is a S_r-subcomplex of $\mathrm{sd}\,B_{\mathrm{edge}}(H)$, we will construct an acyclic partial S_r-matching on $\mathcal{F}(\mathrm{sd}\,B_{\mathrm{edge}}(H))$ whose set of critical elements is $\mathcal{F}(\Delta(i(P_{K_r^r,H})))$.

Note first that, for any chain A of $\mathrm{sd}\,\mathsf{B}_{\mathrm{edge}}(H)$, A is a chain of $\Delta(i(P_{K_r^r,H}))$ if and only if $i \circ p(A_k) = A_k$ for any $k \in [\#A]$. Indeed, if A is a chain of $\Delta(i(P_{K_r^r,H}))$, then we can choose $\varphi_k \in P_{K_r^r,H}$ and write $A_k = i(\varphi_k)$ for each $k \in [\#A]$. Since $p \circ i = 1$, we obtain $i \circ p(A_k) = A_k$. The converse holds by the definitions of i and p.

To achieve our purpose, it suffices to construct an acyclic partial S_r-matching which matches chains not belonging to $\Delta(i(P_{K_2,G}))$. First, we define a subset D of $\mathcal{F}(\mathrm{sd}\,\mathsf{B}_{\mathrm{edge}}(H))$ by
$$D = \{F \in \mathcal{F}(\mathrm{sd}\,\mathsf{B}_{\mathrm{edge}}(H)) \,|\, i \circ p(F_j) \neq F_j \text{ for some } j \in [\#F]\}.$$
$D = \emptyset$ implies that $\Delta(i(P_{K_r^r,H}))$ and $\mathrm{sd}\,\mathsf{B}_{\mathrm{edge}}(H)$ are the same. We assume $D \neq \emptyset$. For any $F \in D$, we let $l(F)$ denote the minimal index l such that $i \circ p(F_l) \neq F_l$, and $r(F)$ the maximal index r such that $F_{l(F)+r}$ is included in $i \circ p(F_{l(F)})$. With these indices, we define $\Sigma_1, \Sigma_2 \subset D$ as follow:
$$\Sigma_1 = \{F \in D \,|\, l(F) + r(F) = \#F,\ i \circ p(F_{l(F)}) \in F\};$$
$$\Sigma_2 = \{F \in D \,|\, l(F) + r(F) < \#F,\ i \circ p(F_{l(F)}) \cap F_{l(F)+r(F)+1} \in F\}.$$
Now we define a map $\mu : \Sigma_1 \cup \Sigma_2 \to \mathcal{F}(\mathrm{sd}\,B_{\mathrm{edge}}(H)) \setminus (\Sigma_1 \cup \Sigma_2)$ as
$$\mu(F) = \begin{cases} F \cup \{i \circ p(F_{l(F)})\} & \text{if } F \in \Sigma_1; \\ F \cup \{i \circ p(F_{l(F)}) \cap F_{l(F)+r(F)+1}\} & \text{if } F \in \Sigma_2. \end{cases}$$
We claim that the pair $(\Sigma_1 \cup \Sigma_2, \mu)$ is an acyclic partial S_r-matching on $\mathcal{F}(\mathrm{sd}\,\mathsf{B}_{\mathrm{edge}}(H))$.

We first check that $\Sigma_1 \cup \Sigma_2$ is an S_r-subset of $\mathcal{F}(\mathrm{sd}\,\mathsf{B}_{\mathrm{edge}}(H))$: let $F = \{F_0, F_1, \dots, F_{\#F}\}$ be an element of $\Sigma_1 \cup \Sigma_2 \subset \mathcal{F}(\mathrm{sd}\,\mathsf{B}_{\mathrm{edge}}(H))$ satisfying $F_0 \subset F_1 \subset \dots \subset F_{\#F}$ and $\sigma \in S_r$. Then $F\sigma = \{F_0\sigma, F_1\sigma, \dots, F_{\#F}\sigma\}$ is a chain of $\mathrm{sd}\,\mathsf{B}_{\mathrm{edge}}(H)$. Since $F \in D$, we can take an index j with $i \circ p(F_j) \neq F_j$. Then S_r-equivariance of $i \circ p$ implies that $i \circ p(F_j\sigma) \neq F_j\sigma$. So $F\sigma \in D$.

Now suppose $F \in \Sigma_1$. The condition $l(F\sigma) + r(F\sigma) = \#F$ holds because of the bijectivity of σ. Since $i \circ p(F_{l(F)}) \notin F$, $(F\sigma)_{l(F\sigma)} = F_{l(F)}\sigma$ and $i \circ p$ is S_r-equivariant, we have $i \circ p((F\sigma)_{l(F\sigma)}) \notin F\sigma$, and so $F\sigma \in \Sigma_1$. Next let $F \in \Sigma_2$. The condition $l(F\sigma) + r(F\sigma) < \#F$ is obvious. The second condition comes from the following calculation:
$$\begin{aligned} i \circ p((F\sigma)_{l(F\sigma)}) \cap (F\sigma)_{l(F\sigma)+r(F\sigma)+1} &= i \circ p(F_{l(F)}\sigma) \cap F_{l(F)+r(F)+1}\sigma \\ &= i \circ p(F_{l(F)})\sigma \cap F_{l(F)+r(F)+1}\sigma \\ &= (i \circ p(F_{l(F)}) \cap F_{l(F)+r(F)+1})\sigma \notin F\sigma. \end{aligned}$$
So $F\sigma \in \Sigma_2$. Summing up, $\Sigma_1 \cup \Sigma_2$ is an S_r-subset.

Next, we must verify that μ satisfies the condition for being a partial S_r-matching: First we find that both $i \circ p(F_{l(F)})$ for $F \in \Sigma_1$ and $i \circ p(F_{l(F)}) \cap F_{l(F)+r(F)+1}$ for $F \in \Sigma_2$ are simplices of $B_{\mathrm{edge}}(H)$.

Hence, $\mu(F)$ is a chain in $\mathrm{sd}\, B_{\mathrm{edge}}(H)$ with relation

$$F_0 \subset \ldots \subset F_{l(F)} \subset \ldots \subset \cdots \subset F_{\#F} \subset i \circ p(F_{l(F)}) \tag{1}$$

for $F \in \Sigma_1$, and

$$F_0 \subset \ldots \subset F_{l(F)} \subset \ldots \subset F_{l(F)+r(F)} \subset i \circ p(F_{l(F)}) \cap F_{l(F)+r(F)+1} \subset F_{l(F)+r(F)+1} \subset \ldots \subset F_{\#F}. \tag{2}$$

for $F \in \Sigma_2$. We can see from the relations (1) and (2) that, for any $F \in \Sigma_1 \cup \Sigma_2$, $\mu(F)$ covers F but is not a chain in $\Sigma_1 \cup \Sigma_2$; moreover, $F_1 \in \Sigma_1$ and $F_2 \in \Sigma_2$ imply that $\mu(F_1) \neq \mu(F_2)$. If we suppose that both F_1 and F_2 belong to Σ_j ($j = 1, 2$) satisfying $\mu(F_1) = \mu(F_2)$, then we find that the inserted terms to obtain $\mu(F_1)$ and $\mu(F_2)$ are in the same index. This yields that $F_1 = F_2$, and so μ is injective. This μ is S_r-equivariant because of the following calculations: if $F \in \Sigma_1$,

$$\begin{aligned}\mu(F\sigma) &= F\sigma \cup \{i \circ p((F\sigma)_{l(F\sigma)})\}\\ &= F\sigma \cup \{i \circ p(F_{l(F)})\sigma\}\\ &= (F \cup \{i \circ p(F_{l(F)})\})\sigma = \mu(F)\sigma.\end{aligned}$$

If $F \in \Sigma_2$, we have

$$\begin{aligned}\mu(F\sigma) &= F\sigma \cup \{i \circ p((F\sigma)_{l(F\sigma)}) \cap (F\sigma)_{l(F\sigma)+r(F\sigma)+1}\}\\ &= F\sigma \cup \{(i \circ p(F_{l(F)}) \cap F_{l(F)+r(F)+1})\sigma\}\\ &= (F \cup \{(i \circ p(F_{l(F)}) \cap F_{l(F)+r(F)+1})\})\sigma = \mu(F)\sigma.\end{aligned}$$

Finally, we find that $\Sigma_1 \cup \Sigma_2 \cup \mu(\Sigma_1 \cup \Sigma_2) = D$, and we can conclude that the pair $(\Sigma_1 \cup \Sigma_2, \mu)$ is a partial S_r-matching on $\mathcal{F}(\mathrm{sd}\, B_{\mathrm{edge}}(H))$ whose set of critical elements is $\mathcal{F}(\Delta(i(P_{K_r^r,H})))$.

It remains to prove that the matching is acyclic: suppose that there exists a sequence of distinct elements $F^0, F^1, \ldots, F^t \in \Sigma_1 \cup \Sigma_2$ $t \geq 1$ such that

$$\mu(F^0) \succ F^1, \mu(F^1) \succ F^2,\ \ldots\ ,\ \mu(F^{t-1}) \succ F^t \text{ and } \mu(F^t) \succ F^0.$$

For each $j \in [t-1]$, since $\mu(F^j)$ covers both F^j and F^{j+1} which are distinct, we can choose a simplex $A_j \in F^j$ such that $F^{j+1} = \mu(F^j) \setminus \{A_j\}$. Similarly, $A_t \in F^t$ can be chosen such that $F^0 = \mu(F^t) \setminus \{A_t\}$.

It is useful if we know what are A_j, $j \in [t]$: we claim here that

$$A_j = \begin{cases} F^j_{l(F^j)} & \text{if } F^j \in \Sigma_1;\\ F^j_{l(F^j)+r(F^j)+1} \text{ or } F^j_{l(F^j)} & \text{if } F^j \in \Sigma_2.\end{cases}$$

In fact, for $F^j \in \Sigma_1$, if A_j were not $F^j_{l(F^j)}$, it follows from the equation (1) that $F^{j+1}_{l(F^{j+1})} = F^j_{l(F^j)}$, and so $i \circ p(F^{j+1}_{l(F^{j+1})}) \in F^{j+1}$; hence $F^{j+1} \notin \Sigma_1$. Since $i \circ p(F^j_{l(F^j)})$ contains all simplices in F^j, we obtain $F^{j+1} \notin \Sigma_2$. Therefore $F^{j+1} \notin \Sigma_1 \cup \Sigma_2$, contradicting to the assumption of F^{j+1}. For $F^j \in \Sigma_2$, if A_j were not $F^j_{l(F^j)}$ and $F^j_{l(F^j)+r(F^j)+1}$, it follows from the equation (2) that $F^{j+1}_{l(F^{j+1})} = F^j_{l(F^j)}$. So $F^{j+1} \notin \Sigma_1$ because the simplex $F^{j+1}_{l(F^{j+1})+r(F^{j+1})+1}$ still exists. Moreover, we obtain $F^{j+1}_{l(F^{j+1})+r(F^{j+1})+1} = F^j_{l(F^j)+r(F^j)+1}$, and then $i \circ p(F^{j+1}_{l(F^{j+1})}) \cap F^{j+1}_{l(F^{j+1})+r(F^{j+1})+1} \in F^{j+1}$. Hence $F^{j+1} \notin \Sigma_2$. Summing up, $F^{j+1} \notin \Sigma_1 \cup \Sigma_2$, which contradicts to the assumption of F^{j+1}.

We can see from the above remark on A_j that, if $F^j \in \Sigma_2$, F^{j+1} can be a chain in either Σ_1 or Σ_2, while, if $F^j \in \Sigma_1$, F^{j+1} can be a chain only in Σ_1 because $i \circ p(F^{j+1}_{l(F^{j+1})})$ contains $i \circ p(F^j_{l(F^j)})$, which contains all F^j_k ($k \in [\#F^j]$). Similarly, $F^t \in \Sigma_1$ implies that $F^0 \in \Sigma_1$. Then we can conclude that there are three cases on a set to which the chains $F^0, \ldots, F^t$ belongs, as follows:

(a) All $F^0, \ldots, F^t$ belong to Σ_1;
(b) All $F^0, \ldots, F^t$ belong to Σ_2;
(c) There exists $j \in [t-1]$ such that $F^j \in \Sigma_2$ but $F^{j+1} \in \Sigma_1$.

We can find a contradiction for the case (c) at once because the fact that $F^k \in \Sigma_1$ whenever $F^{k-1} \in \Sigma_1$ implies that $F^j \in \Sigma_1$. For the case (a), considering the number $t(F^j)$ of indices l such that $F^j_l \neq i \circ p(F^j_l)$, we obtain a contradiction $t(F^0) < t(F^0)$.

For the case (b), let we denote $s(F^j)$ the number of simplices in F^j not contained in $i \circ p(F^j_{l(F^j)})$. By assumption, we have $s(F^j) \geq 1$for any $j \in [t]$. By the assumption, each A_j is $F^j_{l(F^j)}$ or $F^j_{l(F^j)+r(F^j)+1}$. If $A_j = F^j_{l(F^j)}$, the fact that $i \circ p(F^{j+1}_{l(F^{j+1})}) \supset i \circ p(F^j_{l(F^j)})$ implies that $s(F^{j+1}) \leq s(F^j)$. If $A_j = F^j_{l(F^j)+r(F^j)+1}$, then we have $s(F^{j+1}) = s(F^j) - 1 < s(F^j)$. Summing up, $F^0, \dots, F^t \in \Sigma_2$ implies the following inequalities:

$$s(F^0) \leq s(F^t) \leq \cdots \leq s(F^1) \leq s(F^0). \tag{3}$$

We will get a contradiction if there exists a “less than or equal to” sign which is really the “less than” sign. We obtain the assertion at once if there is $j \in [t]$ with $A_j = F^j_{l(F^j)+r(F^j)+1}$.

Assume that $A_j = F^j_{l(F^j)}$ for all $j \in [t]$. By definition, we can choose $F^{j+1}_{l(F^{j+1})}$ and $F^{j+1}_{l(F^{j+1})+r(F^{j+1})+1}$ in each F^j. However, we will get a contradiction

$$F^{j+1} \ni i \circ p(F^{j+1}_{l(F^{j+1})}) \cap F^{j+1}_{l(F^{j+1})+r(F^{j+1})+1}$$

if there exists $j \in [t]$ such that either of these conditions holds:

(c1) $i \circ p(F^j_{l(F^j)}) \cap F^j_{l(F^j)+r(F^j)+1} = i \circ p(F^{j+1}_{l(F^{j+1})}) \cap F^{j+1}_{l(F^{j+1})+r(F^{j+1})+1}$, or
(c2) $i \circ p(F^{j+1}_{l(F^{j+1})}) \cap F^{j+1}_{l(F^{j+1})+r(F^{j+1})+1}$ is distinct from $F^j_{l(F^j)}$ and is in F^j.

Then we can assume that all $j \in [t]$ do not satisfy both conditions. Suppose that $s(F^0) = s(F^1) = \cdots = s(F^t)$. We find that $F^0_{l(F^0)+r(F^0)+1}$ is the minimal simplex not included in $i \circ p(F_{l(F^j)})$ for any $j \in [t]$. Paying attention to the simplices inserted to each chain, we find by our assumption that

$$\begin{aligned} & i \circ p(F_{l(F^0)}) \cap F_{l(F^0)+r(F^0)+1} \subsetneq i \circ p(F_{l(F^1)}) \cap F_{l(F^0)+r(F^0)+1} \subsetneq \cdots \\ & \subsetneq i \circ p(F_{l(F^t)}) \cap F_{l(F^0)+r(F^0)+1} \subsetneq F_{l(F^0)+r(F^0)+1}. \end{aligned} \tag{4}$$

Since $F_{l(F^0)+r(F^0)+1}$ is the minimal simplex not included in $i \circ p(F_{l(F^0)})$, we obtain

$$i \circ p(F_{l(F^t)}) \cap F_{l(F^0)+r(F^0)+1} \subset i \circ p(F_{l(F^0)}).$$

Then,

$$i \circ p(F_{l(F^t)}) \cap F_{l(F^0)+r(F^0)+1} \subset i \circ p(F_{l(F^0)}) \cap F_{l(F^0)+r(F^0)+1}.$$

With (4), we thus obtain a contradiction $i \circ p(f_{l(F^0)}) \cap f_{l(F^0)+r(F^0)+1} \subsetneq i \circ p(f_{l(F^0)}) \cap f_{l(F^0)+r(F^0)+1}$. Therefore, in (3), there exists a “less than or equal to” sign which is really the “less than” sign, and so we get a contradiction $s(F^0) < s(F^0)$.

Summing up, our argument contradicts itself if we suppose that $(\Sigma_1 \cup \Sigma_2, \mu)$ is not acyclic. $\square$

We depict an S_r-collapsing construed by the above acyclic partial S_r-matching for a part of $\operatorname{sd} \mathsf{B}_{\mathrm{edge}}(H)$, $H = K^3_{2,2,1}$ as the following figure. Here we draw a hypergraph by edge-based drawings, see [KKS09].

We now complete our argument in all steps, obtaining a construction of a formal S_r-deformation between $\mathrm{Hom}(K^r_r, H)$ and $\mathsf{B}_{\mathrm{edge}}(H)$. So the following conclusion holds:

Theorem 4.11. For an r-graph H, the Hom complex $\mathrm{Hom}(K^r_r, H)$ and the box complex $\mathsf{B}_{\mathrm{edge}}(H)$ have the same simple S_r-homotopy type.

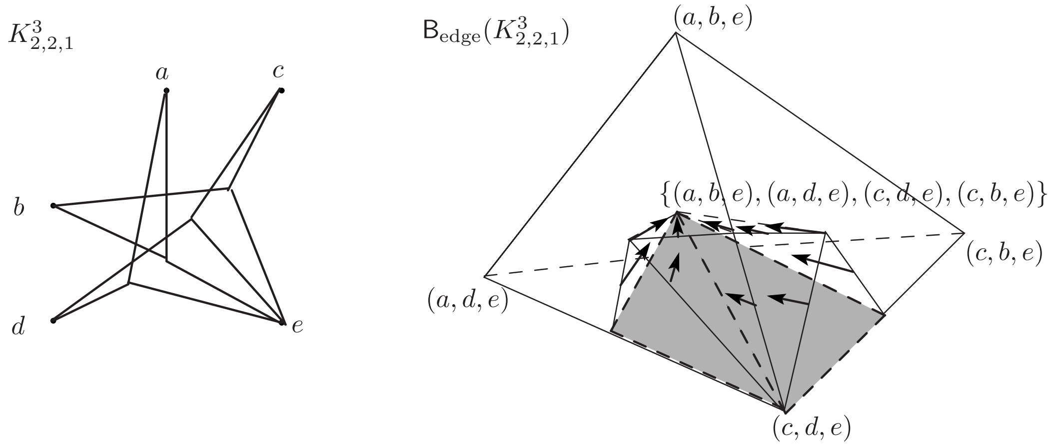

FIGURE 1. $K^3_{2,2,1}$ and a part of the S_3-collapsing of sd $\mathsf{B}_{\mathrm{edge}}(K^3_{2,2,1})$ onto $\Delta(i(P_{K^3_3,K^3_{2,2,1}}))$.

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